Symmetries of magnetic fields: What are they good for?

Outline

PART 0*: Review + Intuition + Examples

PART I : A Noether-type theorem

PART II : Integrable magnetic fields

a) Topology

b) Dynamics

PART 0*

Review + Intuition + Examples

Review

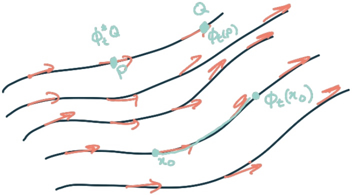
- Differential forms : $Q \in \Omega^k(M)$ (k-form)

  – Encode information like projected area, density or volume, length.

- Pullback of Q by $\phi : M \to M$: $\phi^* Q$

  – Q in new coordinates ϕ

- Flow of a vector field X : ϕ_t



- $\phi : M \to M$ is a symmetry of Q if

$$\underbrace{\phi^* Q}_{\text{Q in new coordinates given by } \phi} = \underbrace{Q}_{\text{Q in old coordinates}}$$

Q does not change under coordinate transform $\phi : M \to M$

- The Lie derivative

$$\boxed{\mathcal{L}_X Q := \frac{d}{dt}\Big|_{t=0} \Phi_t^* Q}$$

- Measures how Q changes (infinitesimally) along the flow of X.

- X is an (infinitesimal) symmetry of Q if:

$$\mathcal{L}_X Q = 0 .$$

Q does not change along X.

Examples

OBJECT	EXAMPLE	$\mathcal{L}_X$ In $\mathbb{R}^3$
Vectors	B	$\mathcal{L}_X B = X \cdot \nabla B - B \cdot \nabla X$
0-forms	f	$\mathcal{L}_X f \simeq X \cdot \nabla f$
1-forms	$B^\flat = B \cdot dl$	$\mathcal{L}_X B^\flat \simeq (\nabla \times B) \times X + \nabla(X \cdot B)$
2-forms	$\beta(u,v) = B \cdot (u \times v)$	$\mathcal{L}_X \beta \simeq \nabla \times (B \times X) + (\nabla \cdot B) X$
3-forms	$\Omega = \rho\, d^3x$	$\mathcal{L}_X \Omega \simeq \rho \nabla \cdot X + X \cdot \nabla \rho$

Symmetry Examples

1) If $J \times B = \nabla p$, $J = \nabla \times B$, $\nabla \cdot B = 0$

$$\mathcal{L}_J B = \nabla \times (B \times J) + \cancel{(\nabla \cdot B) X}$$

$$= -\cancel{\nabla \times (\nabla p)} = 0$$

2) u is a Quasisymmetry of B if :

i) $\mathcal{L}_u B^b = 0$

ii) $\mathcal{L}_u \beta = 0$

iii) $\mathcal{L}_u |B|^2 = 0$

A useful Lemma

Lemma: If $\nabla\cdot X = 0$ then

$$\mathcal{L}_X B = 0 \iff \mathcal{L}_X \beta = 0.$$

Proof (in $\mathbb{R}^3$):

$$\begin{aligned}\mathcal{L}_X B &= X\cdot\nabla B - B\cdot\nabla X \\ &= \underbrace{\nabla\times(B\times X) + (\nabla\cdot B)X}_{\simeq\, \mathcal{L}_X \beta} - (\nabla\cdot X)B\end{aligned}$$

So, if $\nabla\cdot X = 0$ then $\mathcal{L}_X B = 0 \iff \mathcal{L}_X \beta = 0.$ $\square$

Differential Forms Propaganda.

The only identities for differential forms required here are:

1) $\mathcal{L}_X Q = \iota_X dQ + d\iota_X Q$

2) $\iota_{\mathcal{L}_X B} Q = \mathcal{L}_X \iota_B Q - \iota_B \mathcal{L}_X Q.$

3) $d^2 Q = 0.$

ι_X = inner product ← generalisation of dot product & (cross) product.

d = exterior differentiation. ← curl, div, ∇

Want to learn more? : Stephanie Singer, "Symmetry in Mechanics: A gentle, modern introduction", 2004.

From here on: Assume B is a non-vanishing magnetic field:

1) $|B| \neq 0$ everywhere.

2) $\mathrm{div}(B) = \nabla \cdot B = 0$.

PART I

A Noether-type theorem

Noether's Theorem

To each symmetry there is a corresponding conserved quantity

To each conserved quantity there is a corresponding symmetry.

Q: Is there a Noether-type theorem for magnetic fields?

Div-free symmetry $\Longrightarrow$ conserved quantity.

Lemma 2: Suppose $\mathrm{div}(X)=0$. Then

$$\mathcal{L}_X B = 0 \iff \nabla\times(B\times X) = 0$$

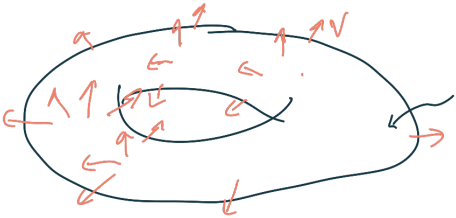
Proof: $\mathcal{L}_X B = 0 \iff \mathcal{L}_X \mathcal{B} = 0 \iff \nabla\times(B\times X) = 0$ $\square$

Lemma 1

Recall: $\nabla\times V = 0 \Longrightarrow V = \nabla\varphi$ in any simply connected domain

no holes.

Fact*: $\nabla \times V = 0 \Rightarrow V = \nabla \varphi$ if V is in a compact domain D of $\mathbb{R}^n$ and normal to the boundary of D.



If X, B tangent to boundary of D,

then $X \times B$ is normal to boundary of D...

* Prop 3.11 of D., Perrella, Pfefferle, "Global realisation of magnetic fields as 1.5D Hamiltonian Systems", 2024

Thm: If 1) $\nabla \cdot X = 0$

2) $\mathcal{L}_X B = 0$

3) X, B tangent to boundary of a solid torus,

then $\exists \psi$ s.t. $X \times B = \nabla \psi$.

Consequently, $\mathcal{L}_X \psi = X \cdot \nabla \psi = 0$

$\mathcal{L}_B \psi = B \cdot \nabla \psi = 0$

So that ψ is a conserved quantity of B.

Remarks

1) By Lemma 2, $J \times B = \nabla p \Rightarrow \nabla \times (J \times B) = 0$

$\Rightarrow \mathcal{L}_J B = 0$

So J is a symmetry of B.

2)* Conditions for QS are $\mathcal{L}_u B^\flat = \mathcal{L}_u \beta = \mathcal{L}_u |B| = 0$

Equivalently, instead of $\mathcal{L}_u \beta = 0$, we can require $\mathcal{L}_u B = 0$ (Lemma 1)

or $\exists \psi$ s.t. $u \times B = \nabla \psi$

*For more mathematics of QS using differential forms:

Burby, Mackay, Kallinikos, "Some mathematics for quasisymmetry", 2020

Conserved Quantity $\psi \Rightarrow$ Symmetry ?

<u>Example:</u> • Linear force-free field $\nabla \times B = \lambda B$

OR Vacuum field $\nabla \times B = 0$.

• Suppose B also has flux surfaces:

$\exists \psi$ s.t. $\mathcal{L}_B \psi = 0$

<u>Q</u>: Does there always exist a symmetry X of B ?

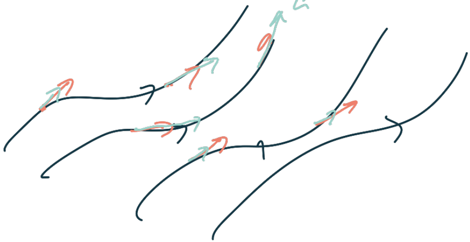
<u>Ans</u>: No! Kind of no...

Conformal symmetry of B

<u>Def</u>: X is a conformal symmetry of B if there exists a strictly positive function ρ s.t.

$$\mathcal{L}_X(\rho^{-1}B) = 0$$

<u>Idea</u>:



$\leftarrow X$ is a symmetry of the rescaling.

Conserved quantity ψ $\Longrightarrow$ Conformal symmetry

Thm*: Suppose there exists a vector field N s.t. i) $N \cdot B > 0$ ii) $(\nabla \times N) \cdot \nabla\psi = 0$

then the field $X = \dfrac{N \times \nabla\psi}{N \cdot B}$ is a conformal symmetry of B with $\rho = N \cdot B$.

Prop*: If ψ defines regular nested tori ($\nabla\psi \neq 0$ on tori) then N always exists.

* See:

Perrella, D., Pfefferle, "existence of global symmetries of divergence-free fields with first integrals", 2023

Example

If $J \times B = \nabla p$ then

$$\text{i) } B \cdot B = |B|^2 > 0$$

$$\text{ii) } (\nabla \times B) \cdot \nabla p = J \cdot \nabla p = 0$$

Taking $N = B$ in Thm gives

$$X = \frac{B \times \nabla p}{|B|^2}$$

as a conformal symmetry of B

PART II

Integrable magnetic fields

Integrable magnetic fields

Def: B is an integrable magnetic field if there exists X, ψ, ρ, s.t.

i) $\mathcal{L}_X B = 0$

ii) $\nabla \cdot (B) = 0 = \nabla \cdot (\rho X)$

iii) $\mathcal{L}_B \psi = 0 = \mathcal{L}_X \psi$

iv) $\nabla \psi \neq 0$ almost everywhere.

Examples

1) (B, J, p) is integrable for $J \times B = \nabla p$

2) $(B, u, 2p)$ is integrable for a QS.

3) $\left(\frac{B}{|B|^2}, \frac{B \times \nabla p}{|B|^2}, p\right)$ is integrable*

* In general: $\nabla \cdot \left(\frac{B}{|B|^2}\right) \neq 0$

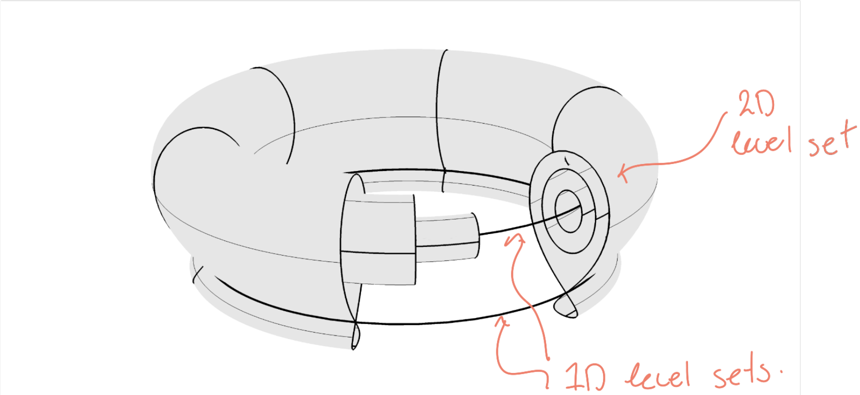
However, with $\rho = |B|^2$ have

$$\nabla \cdot \left(\rho \frac{B}{|B|^2}\right) = \nabla \cdot B = 0$$

& $$\nabla \cdot \left(\rho \frac{B \times \nabla p}{|B|^2}\right) = \nabla \cdot (B \times \nabla p) = 0$$

Topology of Integrable magnetic fields

- $\mathcal{L}_X \Psi = 0 = \mathcal{L}_B \Psi \implies X$ and B lie on level sets of Ψ.



- If level set of 2p is 2D and compact (no bdry) and B non-vanishing

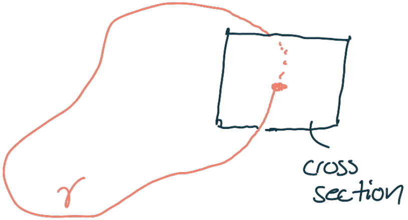
Poincaré-Hopf $\Rightarrow$ 2p is either torus or ~~Klein bottle~~

• If γ is a 1D level set and non-degenerate

$(\nabla_\perp^2 \psi|_\gamma \neq 0)$

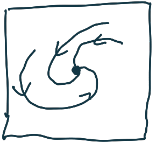
the condition $\nabla \cdot B = 0$ restricts the behaviour of B around γ.

We can look at cross-sections of γ to see the possibilities.



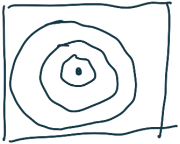
Possibilities of cross-sections :

(un)stable focus



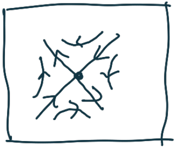
✗ Impossible as volume contracts

elliptic (o-point)



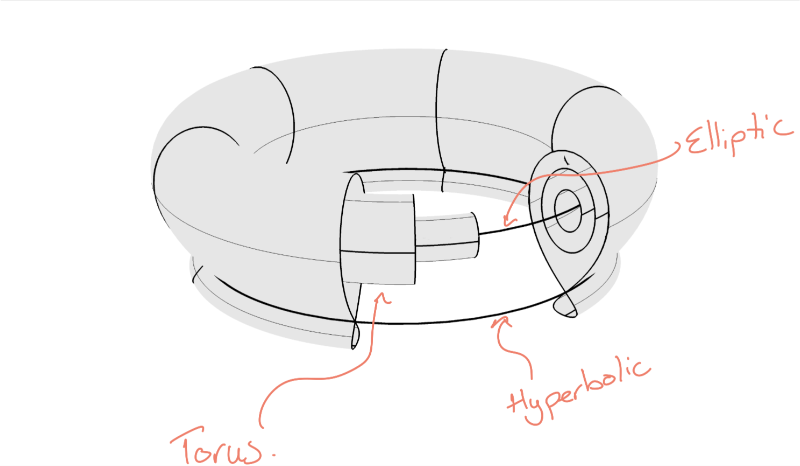
✓ Preserves Volume.

hyperbolic (x-point)



✓ Preserves Volume *if* contraction = expansion

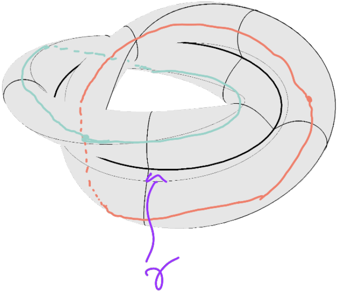
Example 1 : Divertor

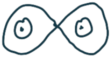


Example 2

a) Hyperbolic

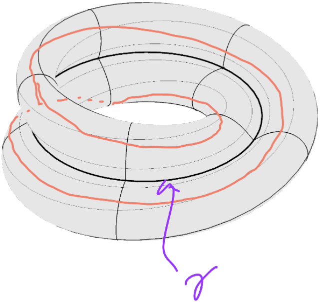
1 - twist





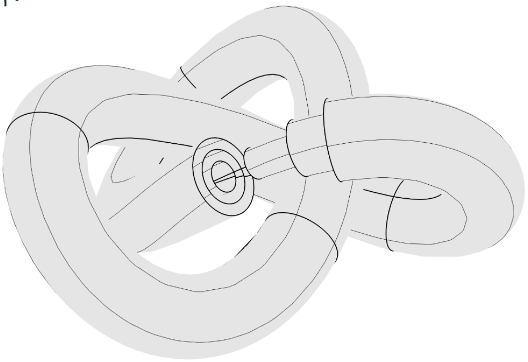
b) reversed hyperbolic.

$\frac{1}{2}$ twist.

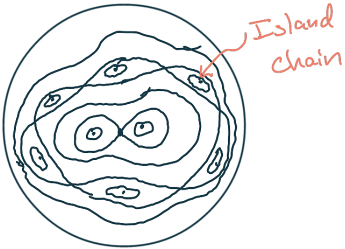


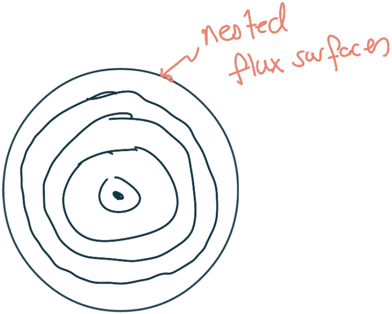
Note: Nested tori $\not\Rightarrow$ simple in $\mathbb{R}^3$.

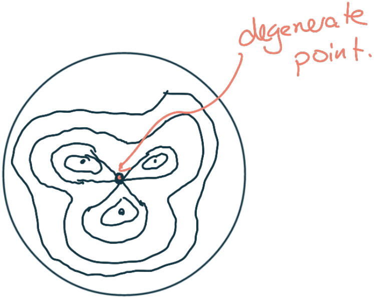
Knototron?

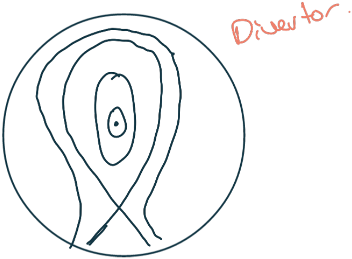


Example Cross-sections.







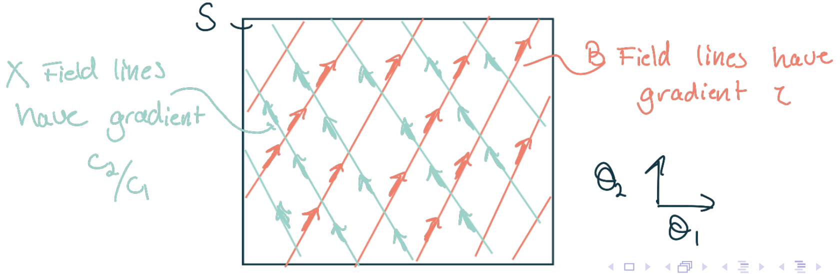


Dynamics of Integrable Magnetic Fields.

Regular tori: level set S where $\nabla\psi \neq 0$.

Thm (Kolmogorov): There exists coordinates $(\theta_1, \theta_2) \in S$ s.t.

1) $B = (1, \iota)$ is a constant vector.
2) $X = (c_1, c_2)$ is a constant vector.



Intuition behind Thm

1) If $\nabla\phi \neq 0$ then B & X preserve area on S.

2) On S, any linear combination of B & X

$$aX + bB$$

is a symmetry of B : $\mathcal{L}_{aX+bB} B = a\mathcal{L}_X B + b\mathcal{L}_B B$

$$= 0$$

3) You can find two pairs (a_1, b_1), (a_2, b_2) so that

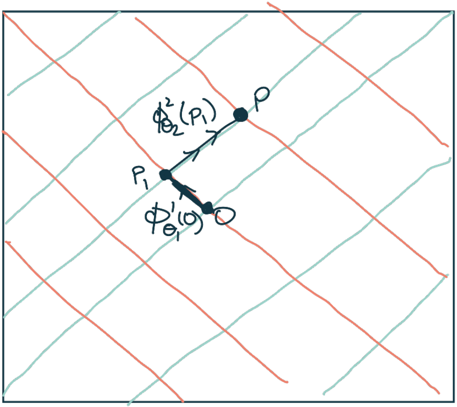
$$\tilde{X}_1 = a_1 X + b_1 B \quad , \quad \tilde{X}_2 = a_2 X + b_2 B$$

are periodic and linearly independent.

4) Use the flow of $\tilde{X}_1$ & $\tilde{X}_2$ to define the coordinates (Θ_1, Θ_2).

Because symmetries.

$$p = \phi_{\Theta_2} \circ \phi_{\Theta_1}(0) = \phi_{\Theta_1} \circ \phi_{\Theta_2}(0)$$



5) Because X & B do not change along $\tilde{X}_1, \tilde{X}_2$ they are constant.

Using similar ideas (see Arnol'd Structure Thm*)
it can be shown these constant field line coordinates
in a neighbourhood of the torus S.

This gives Hamada coordinates. $(\theta_1, \theta_2, \psi)$

$$B = (1, \iota(\psi), 0)$$

$$X = (a_1(\psi), a_2(\psi), 0)$$

constant on each torus

Hamada coordinates are coordinates in which both B and symmetry X are constant on each torus.

Remark If we have a conformal symmetry X:

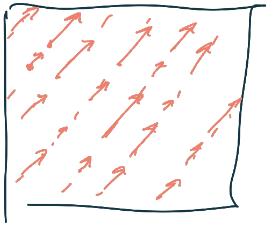
$$(p^{-1}B, X, \partial_p)$$

then Hamada coordinates gives:

$$X = (a_1(\partial_p), a_2(\partial_p), 0)$$

$$B = p \cdot (1, \tau(\partial_p), 0)$$

In thes coordinates, B is straight but changes magnitude with p.



Example: Boozer coordinates.

Boozer coordinates are Hamada coordinates for $\left(\frac{B}{|B|^2}, \frac{B \times \nabla p}{|B|^2}, \nabla p \right)$

Boozer coordinates are coordinates where $\frac{B}{|B|^2}$ and $\frac{B \times \nabla p}{|B|^2}$ is constant.

Example: Boozer for force-free.

If $\nabla \times B = \lambda B$ and we have nested flux surfaces given by ψ, then:

i) There is N with $N \cdot B > 0$
$(\nabla \times N) \cdot \nabla \psi = 0$

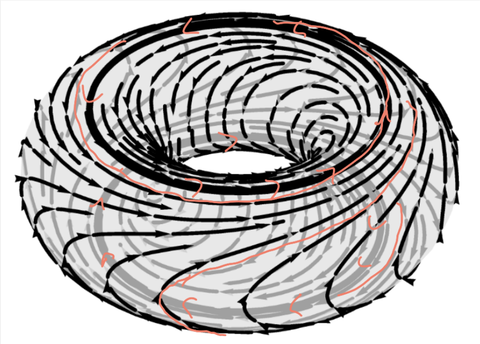
ii) $X = \dfrac{N \times \nabla \psi}{N \cdot B}$ is a conformal symmetry.

iii) There are coordinates where $\dfrac{B}{N \cdot B}$ and X are constant on each torus.

If $\nabla \times B = \lambda B$ can take $\underline{N = B}$.

Dynamics on degenerate tori

- Dynamics can get funky...
- Not classified in general.
- Example: Reeb cylinders



*Reeb cylinders are impossible for $J \times B = \nabla p$.

Dynamics near non-degenerate axis

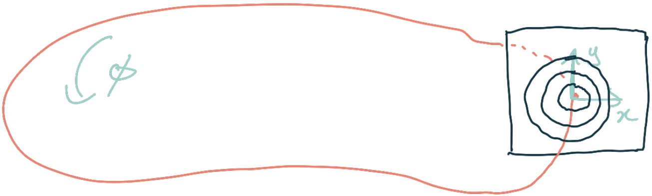
<u>Elliptic</u> : There exist near-axis Hamada coordinates (x, y, ϕ) so that

$$B = \left(-y\, a_1(x^2+y^2)\,,\ x\, a_1(x^2+y^2),\ a_2(x^2+y^2)\right)$$

$$X = \left(-y\, c_1(x^2+y^2)\,,\ x\, c_1(x^2+y^2),\ c_2(x^2+y^2)\right)$$

$$\psi = \psi(x^2+y^2)$$

a_1, a_2, c_1, c_2 are smooth functions, $a_2(0) \neq 0$.



Dynamics near non-degenerate axis

Direct hyperbolic*

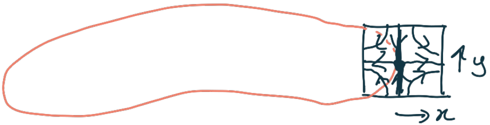
There exist "X-coordinates" (x, y, ϕ) so that

$$B = (x\, a_1(xy), -y\, a_1(xy), a_2(xy))$$

$$\mathcal{P} = \mathcal{P}(xy)$$

$$X = ??$$

$a_1, a_2, \mathcal{P}$ smooth functions, $a_1(0) \neq 0$, $a_2(0) \neq 0$



*See: Burby, D., Meiss, "Integrability, normal forms, and magnetic axis coordinates", 2021

Summary

- Div-free symmetry $\Longrightarrow$ conserved quantity
- conserved quantity $\Longrightarrow$ conformal symmetry
- Integrable magnetic fields live on level sets of conserved quantity. There are generally
  1) Tori
  2) elliptic, hyperbolic axes.
- Symmetry gives rise to <u>nice</u> coordinates near tori or axis that show the dynamics is "simple".

THANK YOU